Raising Turkeys:
An Extension and Devastating Application of Myerson-Weber Voting Equilibrium

Or:
Why Most, but Not All, Theoretically Possible Democratic Institutions are Useless and
Why Most, But Not All, Social Choice Research on Them Has Been (Mostly) Useless

Burt L. Monroe
Michigan State University

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What follows is not a paper, as such, but slides for the presentation (giving highlights of the argument) and an extract from a forthcoming book (giving technical details).

We Have Invented Many Voting Systems

For single winner only these include ...

- Plurality rule
- Runoff & variants
- Alternative vote / instant runoff

- Anti-Plurality Rule
- Borda (1784)
- Scoring / points systems
- Approval Voting (Brams and Fishburn 1978)
- Approval-Disapproval Voting (Felsenthal 1989)

- Copeland (1951)
- Simpson / Kramer / heart (Simpson 1969; Kramer 1974; Schofield 1993)
- Kemeny (1959)
- Slater (1961)
- Dodgson (1876)
- Black (1958)
- Second-order Copeland (Bartholdi, Tovey and Trick 1989)
- Kendall-Wei / eigenvector (Kendall 1955; Wei 1952)
- r^{th} - order versions
- Coombs / exhaustive voting (1954)
- Nanson (1882)
- Elimination versions
- Converse consistent versions
- Voice of Reason (Monroe)
- Voter elimination versions

Don Saari's Beverage Parable

Fifteen mathematicians buying drinks for a party

6: Milk > Wine > Beer
5: Beer > Wine > Milk
4: Wine > Beer > Milk

(Put aside that this is a PR, not winner-take-all problem ⇒ Buy milk, beer, and wine in proper proportions, and don't invite me — sheesh.)

- Plurality ⇒ Milk
- Runoff ⇒ Beer
- Pairwise Comparisons (Condorcet) ⇒ Wine
- Borda Rule ⇒ Wine

Saari likes Borda because it has symmetry:

- It treats voters equally
- It treats candidates equally
- It treats preferences equally down the preference order

An Arrow's Theorem Primer

May's Theorem

When choosing between two alternatives, if we want:

- *Anonymity* (treat all voters equally)
- *Neutrality* (treat both candidates / alternatives equally)
- *Universal Domain* (no limit on preferences)
- *Positive Responsiveness* (if we have a tie, a preference switch breaks it)

==> Majority rule is the only rule.

When choosing among three or more, we also want:

- *Independence of Irrelevant Alternatives* (IIA) (The social ranking of any pair should depend ONLY on individual rankings over that pair.)
  Equivalently: if we prefer A to B, we do so whether or not C is an option

May and IIA ==> Ranking of ANY number must be by majority rule on the pairs.

- *Transitivity* (A beats B and B beats C means A beats C)
  Equivalently: we must always know who wins.

But, remember Condorcet, Lewis Carroll, Duncan Black, etc -- majority rule cycles are possible:

Voter 1: A>B>C
Voter 2: B>C>A
Voter 3: C>A>B
=> Majorities: C<B<A<C<B<A<C<B<A<C...

==> Can't have all six of the above at once

Arrow gets a bit cute to flex his math muscle:
Can relax Anonymity to *Nondictatorship*
Can relax {Neutrality, Positive Responsiveness} to *Pareto Optimality*
==> Five weaker conditions, but still impossible.

Is it likely to be a problem?

Any decision that has ANY distributive element (who gets what, I prefer more, you prefer more) has cyclic preferences. There are very few *political* situations where that's not true.

So What? Can we live without one of the axioms?

- Anonymity: Someone has more power than another (necessarily undemocratic?).
- Neutrality: Some idea has more power than another (necessarily undemocratic?).
- Universal Domain: Some idea or preference is ruled out from the beginning (undemocratic?).
- Positive Responsiveness: Opens up different decision rules on pairs (e.g., 2/3 rule), but not much else.
- Transitivity: We will occasionally never know who won (& if we call that a "tie", the rule's transitive).
- IIA: Allows manipulation — Riker: some will lie, we will never know which ones, and outcomes will be arbitrary or to the advantage of the best manipulators.

Which one DO we live without?
Minor violations of all but A, N, U, PR, but **ALWAYS IIA** ==> We must learn to live with manipulation.

Myerson-Weber Voting Equilibria

Roger Myerson and Robert Weber. 1993. *APSR.*

Scoring Rules: Plurality, Borda, Anti-plurality, Approval Voting.

Bayesian Nash approach:
- Mutually optimal voting strategies.
- Mutually consistent beliefs.

Voters condition their vote on there being a "tie" in which it makes a difference in the outcome.

Given the condition of a tie occurring, *beliefs take the form of conditional probabilities of ties* between particular pairs of alternatives, q_{ij}. (So, $\Sigma_{ij}\, q_{ij} = 1$.)

Voters are drawn randomly from a preference distribution (so ties always have a nonzero probability of occurring).

Beliefs satisfy rational expectations in the following sense:
- All voters hold identical beliefs.
- Beliefs must be consistent with an "ordering condition".

Ordering Condition
- Under equilibrium strategies, each alternative j has an expected score, S_j. The alternative(s) with the highest score is (are) the *expected winner*(s).
- $q_{jk} > 0$ only if
    - j and k among the set of likely winners (have the same expected score).
    - one is the only likely winner and the other is among those with the second-lowest expected score.

Nonelection of Irrelevant Alternatives (Nonelection of Turkeys)

A new, incredibly weak, axiom.

A voting system satisfies NIA if there is no [Myerson-Weber] voting equilibrium in which the set of winners includes an alternative that is considered by all voters to be the least preferable.

Working example

- Three candidates: 1, 2, and 3.
- Two types of voters: A and B.
- Utilities as follows:

Type	Fraction	u_1	u_2	u_3
A	0.5	10	τ	0
B	0.5	τ	10	0

where ($0<\tau<10$).

Candidate 3 is on the ballot, but everyone considers him to be the worst possible.

A system violates NIA if 3 is elected in equilibrium.

Scoring Systems

Votes consist of a point vector: e.g., [1, w, 0].

- Plurality rule, $w = 0$
- Anti-plurality rule, $w = 1$
- Borda rule, $w = ½$.

Voters rank alternatives according to *prospective rating*:

$$r_j = \Sigma_{k \neq j}\, q_{jk}(u_j - u_k)$$

That is, an alternative is ranked more highly by a voter if it is likely to be involved in ties with alternatives the voter likes less.

⇒ Plurality: Vote for the highest rated.
⇒ Anti-plurality: Vote against the lowest rated.
⇒ Borda: Rank by prospective rating.

Also works for approval voting.
⇒ Approval: Approve of those with $r_j > 0$.

Paired Comparison Voting Equilibria

Many more systems can be described as paired comparison systems: Copeland, Simpson, Kemeny, Kendall-Wei, Nanson, etc., etc., (Borda).

Need a new Myerson-Weber-like equilibrium concept for such systems.

- *Collective preferences are a matrix*, not a vector.
- *Decisive outcomes* involve more than a clear win for a single alternative (e.g., a cycle under Copeland is a three-way tie).
- *Pivot outcomes* are "ties" between decisive outcomes (e.g., one vote can pivot between 1>2>3, where 1 wins, and 1>2>3>1 or ? , which is a three-way tie, under Copeland).

Paired Comparison Voting Equilibria

- *Voter strategy choices are an expression of preference on each pair.* They consider each pair separately, generating a prospective rating for each member of the pair.

  For example, the decision between 1 and 2 depends on the likelihood of the pivot outcomes where the 1-2 vote is pivotal and utility differential between the decisive outcomes at that pivot.

- New *ordering condition* analogous to Myerson-Weber. The new condition applies on each pairwise vote. Equilibrium still requires strategies and beliefs that are consistent with one another. (Existence is just a fixed-point theorem.)

Bottom Line for Social Welfare and Choice

Roughly speaking, all reasonable social welfare functions, social choice functions, and voting systems can be characterized by their level of *symmetry* and *responsiveness*.

For social welfare functions and social choice functions, where we know preferences and manipulation (and therefore IIA violations) are unimportant, we can have both.

The "best" function depends on how preference is correctly characterized (Borda if ordinal, utilitarianism if cardinal, approval voting if bifurcated, etc.).

Bottom Line for Voting Systems

For voting systems, the presence of manipulation (specifically turkey-raising) means that we cannot have both. Most systems are too symmetric or too responsive or both. They are useless.

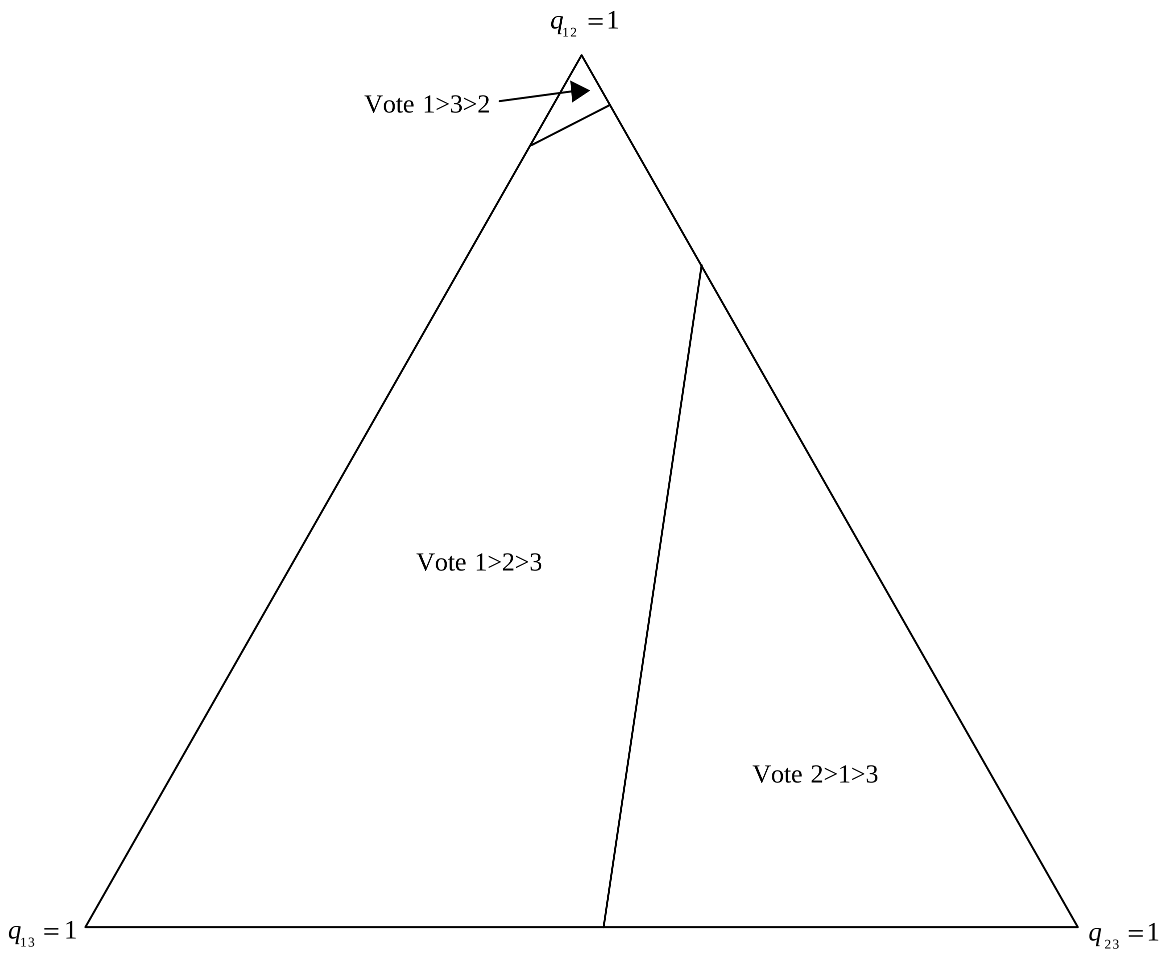
Our institutions must

- make turkey-raising impossible by asking people for only very crude *asymmetric* preference information (plurality rule, approval voting), or
- make turkey-raising ineffective by being *unresponsive* to most preference information (voice of reason), or
- some combination of both (alternative vote, runoff).

Functional twist:

The real world has already figured this out, as it must. All national-level electoral systems in use today are forms of plurality, near-plurality, runoff, alternative vote, and approval voting that meet these criteria. Experiments with others have all eventually failed.

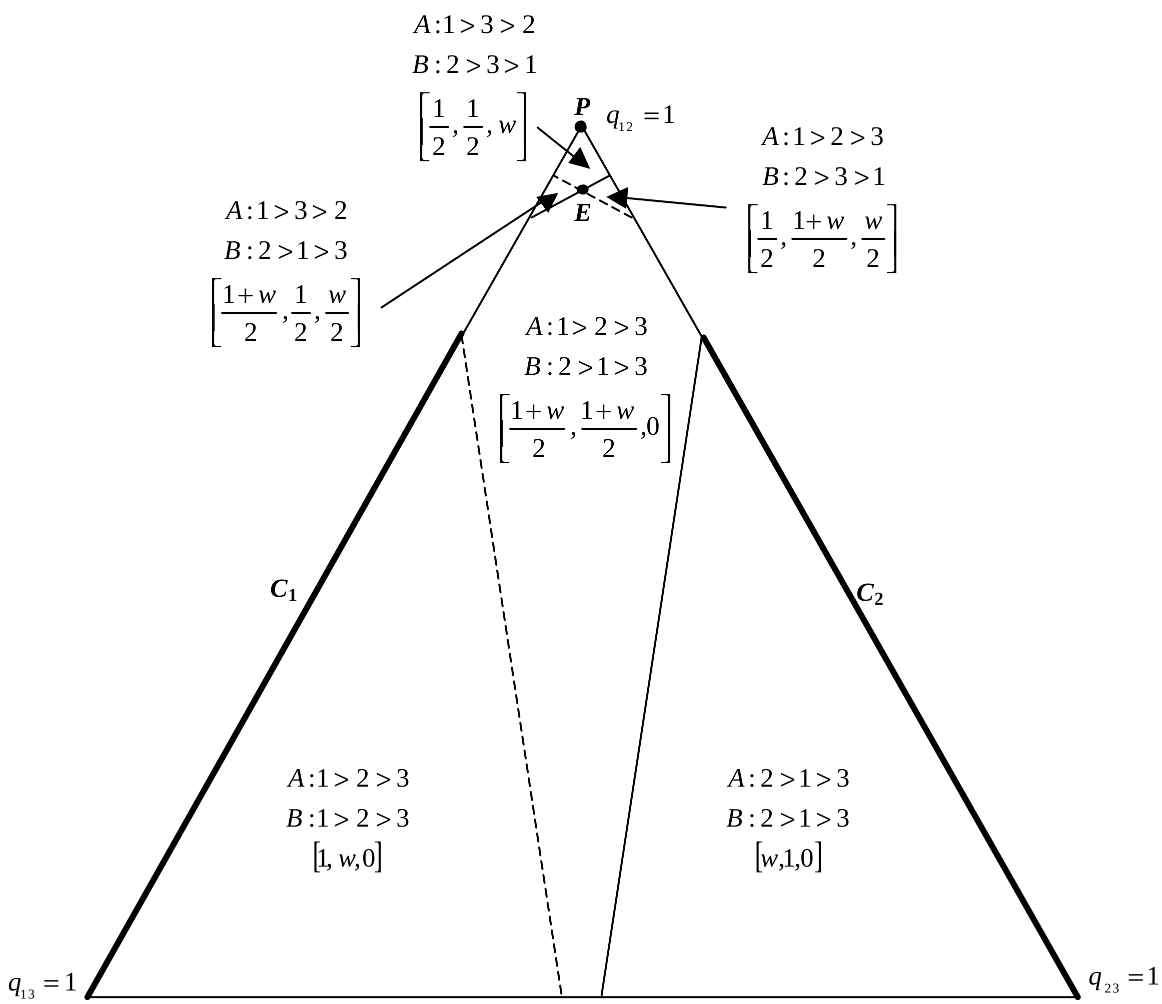
Figure 1 — Strategies and Beliefs for A-type Voters, NIA Example (Scoring Rules)



The triangle represents the simplex of all possible beliefs about conditional tie probabilities, $q_{12} + q_{13} + q_{23} = 1$.

A-type voters have utilities for candidates 1, 2, and 3 as follows. $u_1^A = 10$, $u_2^A = 9$, $u_3^A = 0$.

Figure 2 — Determining Myerson-Weber Equilibria, NIA Example (Scoring Rules)



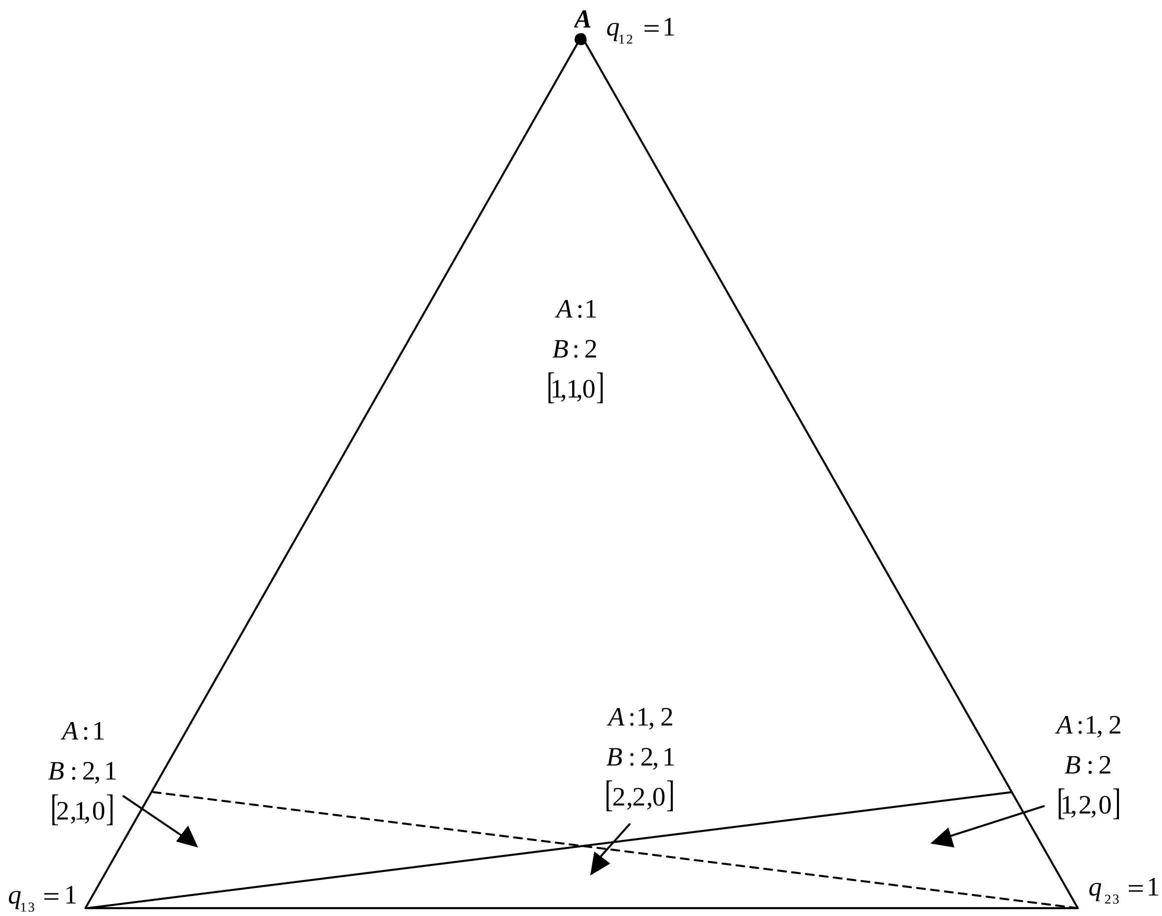
For each region of the belief space, vote strategies are shown for *A* and *B* type voters along with the expected vote scores for candidates 1, 2, and 3 in brackets.

C$_i$: Asymmetric plurality rule equilibria.
All voters vote for alternative *i*.
Alternative *i* wins.

P: Symmetric plurality rule equilibrium and the unique equilibrium for near-plurality rules ($w < 0.5$).
All voters turkey-raise.
Candidates 1 and 2 have the same expected vote score (tie).

E: The unique equilibrium for Borda to anti-plurality ($w \geq 0.5$).
Some of both voter types turkey-raise.
All three candidates have the same expected vote score (tie). [$\Rightarrow$ **NIA is violated.]**

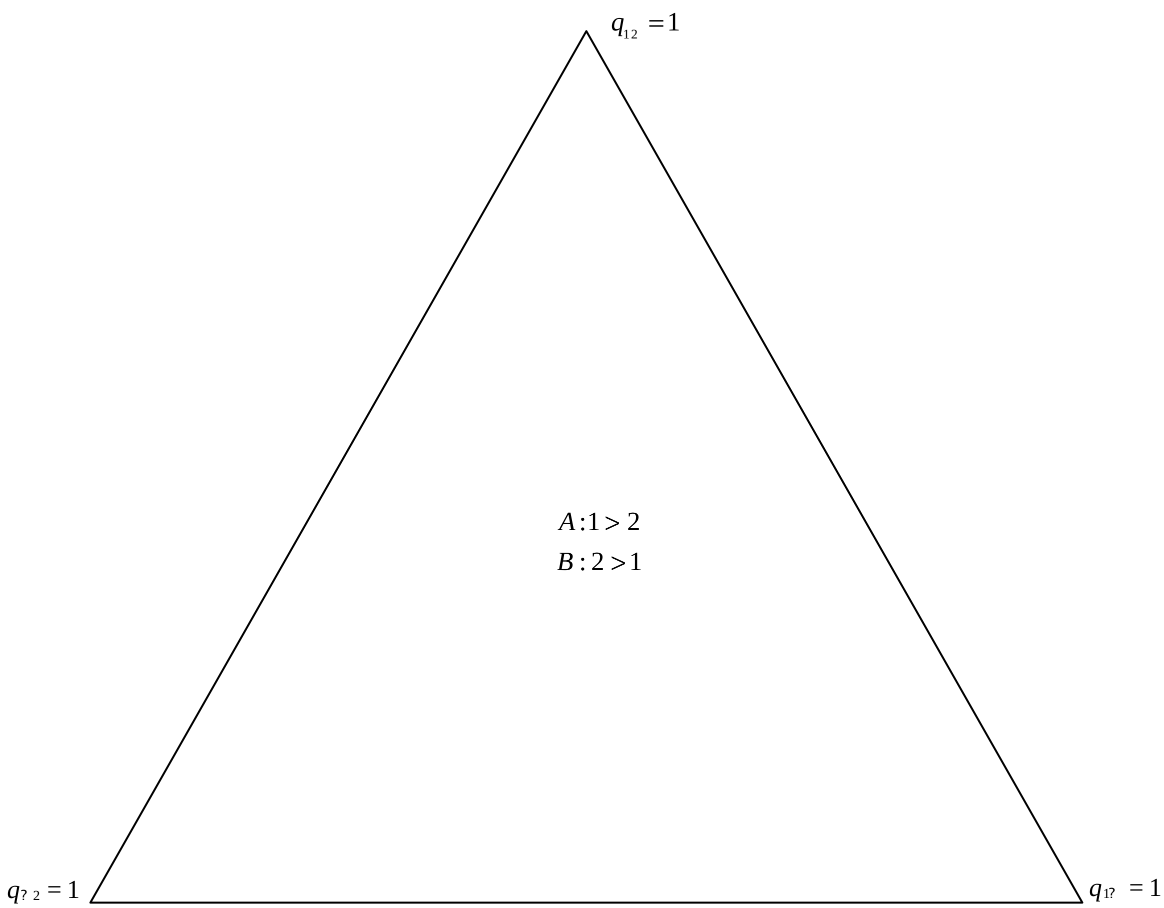
Figure 3 — Determining Myerson-Weber Equilibria, NIA Example (Approval Voting)



For each region of the belief space, vote strategies are shown for *A* and *B* type voters along with the expected vote scores for candidates 1, 2, and 3 in brackets.

A: Unique approval voting equilibrium.
All voters vote for only their first preference.
Candidates 1 and 2 have the same expected vote score (tie).

Figure 4a — Determining Paired Comparison Equilibria, NIA Example (Copeland Rule, Candidates 1 and 2)



For each region of the belief space, vote strategies are shown for *A* and *B* type voters on the pairing of candidates 1 and 2.

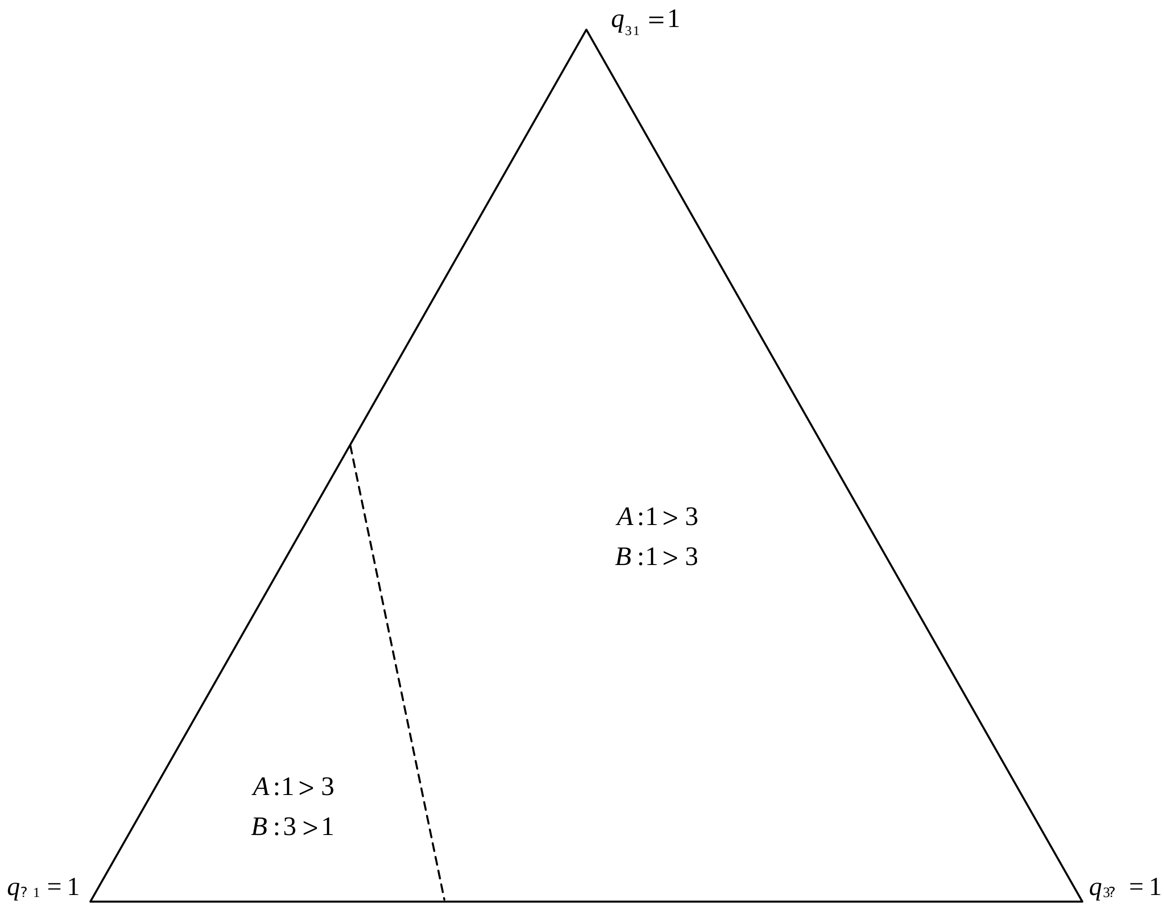
Here, all voters have a dominant strategy to vote sincerely on the 1-2 pair.

Ordering condition:

Sincere voting by all is consistent with only $q_{12} = 1$.

Turkey-raising by all is consistent with all beliefs.

Figure 4b — Determining Paired Comparison Equilibria, NIA Example (Copeland Rule, Candidates 1 and 3)



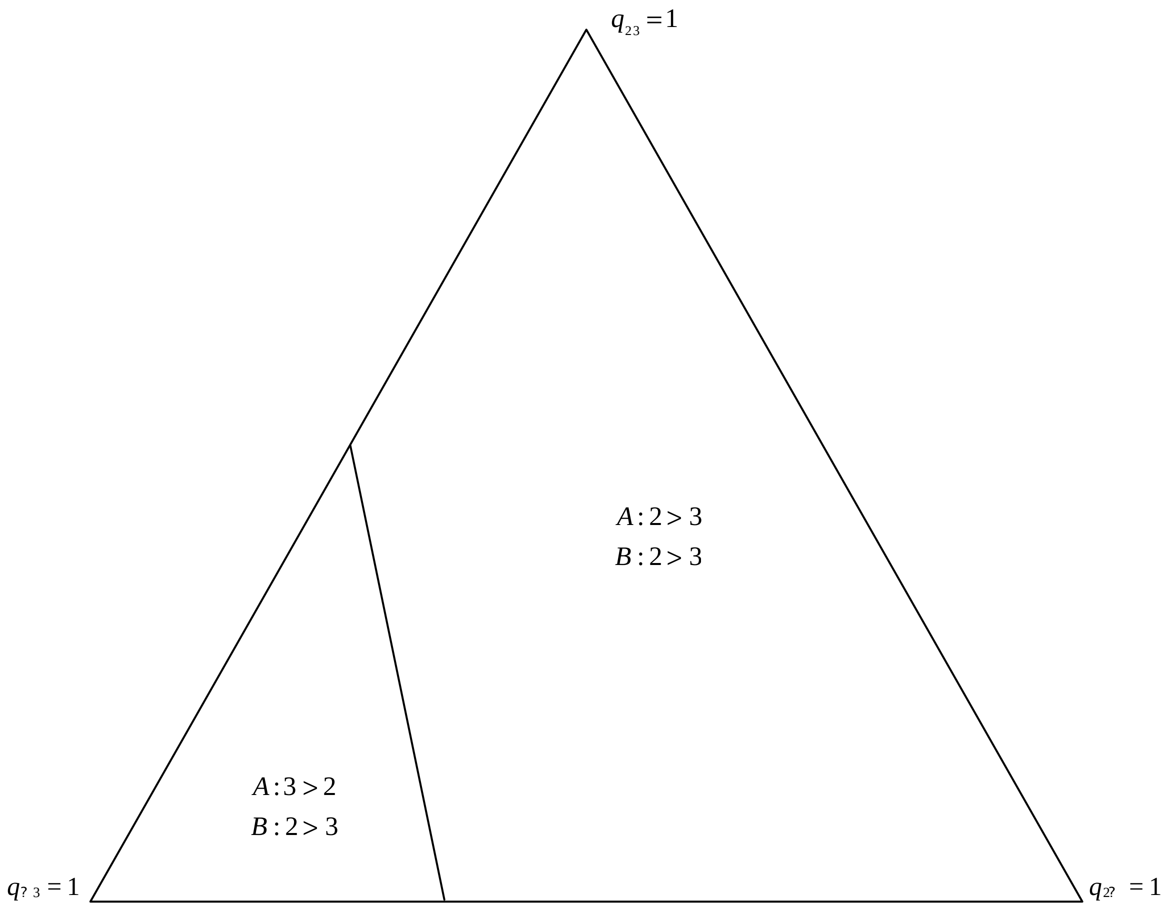
For each region of the belief space, vote strategies are shown for *A* and *B* type voters on the pairing of candidates 1 and 3.

Here, A-type voters have a dominant strategy to vote sincerely on the 1-3 pair, but B-type voters may have an incentive to turkey-raise and vote for 3 over 1.

Ordering condition:

Sincere voting by all is consistent with only $q_{?1} = 1$. (Inconsistent $\Rightarrow$ Not in equilibrium.)
Turkey-raising by all is consistent with all beliefs.

Figure 4c — Determining Paired Comparison Equilibria, NIA Example (Copeland Rule, Candidates 2 and 3)



For each region of the belief space, vote strategies are shown for *A* and *B* type voters on the pairing of candidates 2 and 3.

Here, B-type voters have a dominant strategy to vote sincerely on the 2-3 pair, but A-type voters may have an incentive to turkey-raise and vote for 3 over 2.

Ordering condition:

Sincere voting by all is consistent with only $q_{?3} = 1$. (Inconsistent $\Rightarrow$ Not in equilibrium.)
Turkey-raising by all is consistent with all beliefs.

Relevant Extract from

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Burt L. Monroe
Michigan State University

4.2 Preference Aggregation

Taxonomy is not its own end. The point of the exercise of the last section is (1) to help us be systematic in outlining the "menu" of voting systems, and (2) to help us identify the general characteristics that lead to important system properties. In this section, I try to outline some key preference aggregation properties of multiple-alternative voting systems and the taxonomic characters to which these properties can be traced. In general, the goal in preference aggregation is maximizing social utility (or matching the sincere Borda winner if we know only ordinal preferences), although we may also be interested in the equity of social utility. This differs from the usual axiomatic interests of social choice.

This is very difficult to do generally, of course. We can think of an infinite number of agglomerations of voters and alternatives and preferences. The voting system that can be predicted to maximize utility in one instance may not be the same as in another.

In that light, my first goal here is a basic brush-clearing exercise, although it turns out to clear a great deal of brush. I investigate a superficially very simple property — an axiom if you like — that voting systems must have if there are to be in any way useful for utility maximization. This property is the ability to not select a universally disliked alternative. That is, if we have some alternative that each and every voter puts last in her preference ordering, behind all other alternatives, a voting system should not select it. It turns out that almost all of the systems that have been discussed here — anti-plurality voting, Simpson, Nanson, Copeland, second-order Copeland, Kendall-Wei, Slater, Kemeny, Dodgson, Black, Coombs, *and Borda* among them — fail to have this property. In fact, under these systems and under fairly general conditions, a completely "irrelevant" alternative is as likely to win as any other. If we accept the existence of this property as axiomatic, we are left with very few voting systems that are acceptable.

Cox examines an interrelated phenomenon under the label of "turkey-raising" (Cox 1997: 146-8). Turkey-raising is the tendency of voters to increase the number of votes for candidates at the bottom of their preference orders for strategic gain. The irrelevant alternative in my example is a "turkey" for everyone; it becomes a likely winner under some voting systems because of turkey-raising incentives produced by those systems. Cox's purpose is to note how many candidates (parties, etc.) are competing. Under many familiar voting systems, turkey-raising decentralizes votes and creates more "viable" candidates. My concern is essentially the same, except that I care only about the circumstances under which the turkey can win. If the turkey appears in balloting to receive some support in spite of being last in all sincere preferences, but nevertheless still has no chance of winning, then the system is a success by this criterion. That is, only systems subject to *effective* turkey-raising are considered poor voting systems.

In this section, I first describe the nonstrategic approach to preference aggregation that is standard in the social choice literature, focusing in particular on Saari (1994) and Merrill (1988). I then turn to the strategic approach, building on the Myerson-Weber (1993) and Cox (1997) agenda, expanding the model to apply to more voting systems. Throughout, I will focus on the extent to which voting systems are capable of avoiding selection of irrelevant alternatives (turkeys) as winners. Surprisingly, this will allow us to narrow our focus on viable voting systems in this democratic setting to a very small handful. I conclude the section with a discussion of utility maximization more generally among the handful of systems that do not elect irrelevant alternatives.

4.2.1 How Not to Study Preference Aggregation

I have already argued that the key to understanding an institution is understanding the incentives it gives to those who must live with it. Social choice theory in the Arrovian tradition has been axiomatic, taking preferences as known and given. This is appropriate for the normative task of determining what outcomes should be for any given underlying profile of preferences — the task of determining a social preference function. This is not appropriate when analyzing voting systems. In a voting system, "preferences" are, at best, *expressions of preference* and, at worst, *strategy choices* in a particular voting game.

This is simultaneously *obvious* (to most game theorists) and *objectionable* (to many social choice theorists). Representative of the latter is the discussion of strategic voting by Don Saari in his *Geometry of Voting* (1994). Saari does not address the issue of strategic manipulation until page 262 of the 372-page volume, assuming sincere revelation of preferences in his analysis to that point. By page 267, he has summarized his reasoning for burying the topic: "A prerequisite for strategic action ... is prior knowledge or expectation about the sincere election outcome!" By page 275, his discussion of the topic is complete. Saari is an extremely adept mathematician who cannot be accused of shying away from game theory for technical reasons. His is a substantive objection that, in order to manipulate, voters need a great deal more knowledge than they can generally be expected to have.

Moreover, for Saari, the possibility of manipulation only reinforces the arguments he develops by assuming sincere voting. The universe of voting systems that is the target of most of Saari's analyses are the scoring systems: single-vote plurality (SVP), Borda, anti-plurality voting (APV) and all in between. He also has brief discussions of approval voting and elimination versions of the scoring systems and Copeland. A conclusion common throughout the book is that the property under investigation, whatever it may be, is best delivered by the Borda system and less well-delivered as we stray further away from Borda (to SPV or APV). This is completely parallel to (albeit in greatly more depth and detail than) my axiomatic characterization of Borda in Chapter 2 and its identification as the "fundamental" ordinal preference system earlier in this chapter. If we know preferences — if they are sincerely offered — Borda distorts those preferences the least in aggregating them.

We arrive at a similar conclusion from another perspective by examining the social choice work of another mathematician, Sam Merrill, as presented in his *Making Multicandidate Elections More Democratic* (Merrill 1988). Merrill is concerned with the "Condorcet efficiency" (tendency to pick Condorcet winners when they exist) and "social-utility efficiency" of voting rules. The latter corresponds more or less directly to the utilitarian ideal I have discussed here. The universe of rule he analyzes is SVP, runoff, alternative vote ("Hare"), approval, Borda, Coombs, and Black. His approach is simulation, randomly assigning preferences over a varying number of candidates. He first assumes sincere revelation of preferences, and finds Borda to be the most efficient for a variety of assumptions. Again, this would be expected from knowing it is the least distorting of these rules.[1]

Merrill follows this analysis with a discussion of strategic voting. He considers two possibilities. The first is that voters vote under "uncertainty", having no idea what others' preferences are. His conclusion here is that voters' optimal strategic votes are the same as their sincere votes. Here again, Borda would be the most utility-efficient. The second is that voters vote under "risk", having some sense of others' preferences. He derives optimal voting strategies for plurality, approval voting, and Borda. He concludes, however, that the strategies under SVP and Borda are too complicated to expect voters to actually follow and that one might expect them instead to assume equiprobable outcomes, as under "uncertainty", and therefore to vote essentially sincerely (Merrill 1988: 62).

Both Saari and Merrill can be read, then, as arguing that Borda is an optimal (or at least among the very best) voting system for aggregating preferences. Its optimality (given certain assumptions) when preferences are sincerely revealed is unsurprising. For each of them, the possibility that strategic voting might undermine this optimality is unimportant. Saari cannot imagine voters having the necessary information to vote strategically under any voting system; Merrill cannot imagine voters having the necessary computational abilities (or the desire) to vote strategically under any but the simplest voting systems.

If we accept that voters are sincere, then the system that distorts preferences the least — Borda, utilitarianism, approval voting, etc., depending on our definition of the preference primitives — is ideal. The greater the distortions introduced by points filtering, aggregate processing, and other institutional variations, the greater the degradation of optimal performance. Given the sincerity assumption, I agree wholeheartedly with Saari, Merrill, and others (e.g., Young 1974; Black 1976) that Borda is more or less the best voting system available. I do not, however, accept the assumption voters will (necessarily) be sincere. When we allow for the possibility of strategic manipulation, our judgment of voting institutions changes dramatically. Among other things, we are faced with the conclusion that *most* of the voting systems under discussion here — Borda among them — are *useless*. I turn now to demonstration of this surprising assertion.

4.2.2 Equilibrium Analysis

The best theoretical apparatus yet devised for analyzing the strategic equilibria of voting games is that of Myerson and Weber (1993). This is, with minor modifications, the same apparatus used by Cox in his extensive analysis of strategic voting in real-world electoral systems (Cox 1997). The essence is a Bayesian Nash approach, in which voters must not only find mutually optimal strategies but must hold beliefs that are mutually consistent with those strategy choices. It should also be noted that they do not need the information that Saari insists upon, but rather need only to hold subjective beliefs (that are consistent with strategy choices in equilibrium). These are often justified as the result of preelection

[1] And assuming some symmetry in the random distribution of preferences, as Merrill does.

polling (Myerson and Weber 1993; Forsythe et al. 1993; Fey 1997), but in many voting games, the equilibrium is unique and there is only one thing that voters *can* believe.[2] In any case, I will be using the same approach here to investigate the preference aggregation properties of voting systems.

I do not wish to provide the definition of Myerson-Weber equilibria in its entirety here, but only to sketch it in outline. The reader is referred to Myerson and Weber 1993 for more formal discussion. As in the last chapter, voters must follow the weakly dominant strategy of assuming that their vote matters. Their vote matters only if there is a "tie" of some sort. Voters are drawn randomly from some probability distribution, implying that no matter how lopsided preferences are, there is always a nonzero probability of a tie. Myerson and Weber are concerned only with scoring rules (counting approval voting as a scoring rule), under which a "tie" is just that: a tie in overall vote scores. Unlike the last chapter where we were concerned with only two alternatives, here we must be concerned with *which* two alternatives are in the tie. This is, in fact, the substance of the beliefs in this game: the probability that, given a tie has occurred, it was between any two given alternatives.[3] Specifically, each voter believes that, conditional on a tie having occurred, the probability that alternative j and alternative k are the two who have tied is q_{jk}, where $\sum_{j<k} q_{jk} = 1$.

The driving force underlying Myerson-Weber equilibria is an "ordering condition" on these beliefs. Given the voting strategies in any particular voting equilibrium, each alternative k has an expected vote total ("score"), S_k. The ordering condition requires that if $S_j < S_k$, then $q_{jh} \le q_{kh}$ (for all other alternatives h). That is, voters cannot believe that alternative j is more likely than k to be in a tie for first and then vote in a way that makes it more likely that k is in such a tie.

This has several implications, the most important of which is that $q_{jk} > 0$ only if one of two very specific things is true. First, q_{jk} may be nonzero if both alternative j and alternative k are among the set of likely winners (i.e., have the same expected score). Second, q_{jk} may be nonzero if one of the alternatives is the *only* likely winner and the other is among those with the second-highest scoring group of alternatives. That is, both must be the largest direct threats to one another to win the election.

This is not as complicated as it might seem at first glance. The logic in an SVP election, for example, is by now quite familiar. First, we may have "Duvergerian" equilibria, in which there are only two candidates competitive. Here, a voter who might prefer a candidate other than one of these two must in

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[2] For those that remain uncomfortable with the concept of voters making such strategic calculations, it should be noted that it is a trivial extension of the model to envision elites who make the strategic calculations and then inform voters of them. The strategies tend to be very straightforward — "Smith is the most likely to beat our guy Jones, so Jones-supporters please cast your ballots for Jones then Thomas then Smith" — so this becomes completely plausible. Cox (1997) discusses at length the theoretical and empirical plausibility of such elite-directed strategic manipulation under several electoral systems.

[3] It is assumed that the probability of a three-way tie is infinitesimal compared to the probability of each constituent two-way tie, and can be ignored.

equilibrium abandon that candidate in favor of their favorite between the two leaders. That is, if voters believe only two candidates are competitive, they must vote for one of those two, sustaining the belief. The belief and the behavior are consistent with one another. There are also "nonDuvergerian" equilibria, in which one candidate is clearly leading, and two others are competitive for second place. Here, a voter who might prefer a candidate other than one of these three must in equilibrium abandon that candidate. Three candidates remain competitive because there is unresolved confusion about which of the two trailing candidates is the best competitor to the leader; under no circumstances can a fourth be competitive (see also Cox 1994, 1997; Fey 1997).

To illustrate their model and compare the three different voting systems, Myerson and Weber examine two specific examples.[4] The first is one in which a 60% majority is split between two candidates, while a 40% minority is united behind a third. The interesting issue here is whether the majority can resolve its coordination problem to defeat the minority (and maximize the overall utility). They find that plurality rule has three equilibria: two in which the majority voters do coordinate behind a single candidate and one in which they do not and the minority candidate wins. They find that approval voting also has three equilibria, but that the minority candidate is not a likely winner in any of them. They find that the Borda rule has infinitely many equilibria, all very similar, in which *all three* are likely winners. In this example, then approval voting is unique in maximizing utility, plurality voting *may* maximize utility (depending on equilibrium selection and belief formation), and the Borda rule will not maximize utility.

The second example is similar, except that the minority has a support of only 2% of voters, with the other 98% split between the two major candidates. Here, they find that approval voting yields a unique equilibrium in which voters vote only for their favorite candidate (maximizing utility). Plurality rule also has this as an equilibrium, but additionally has equilibria in which majority voters coordinate on a single candidate (still maximizing utility). Borda rule, on the other hand again has infinitely many equilibria, all of which have all candidates — including the candidate that is the first choice of only 2% of the electorate — as likely winners (not maximizing utility). So again, approval voting and plurality rule *will* maximize utility, and the Borda rule will not.

4.2.3 The Nonelection of Irrelevant Alternatives

Having dismissed Independence of Irrelevant Alternatives (**IIA**) in earlier chapters, I now wish to use the Myerson-Weber strategic voting equilibrium — generalized to other sorts of electoral systems below — to introduce a new axiom: Nonelection of Irrelevant Alternatives (**NIA**). Voting system S fails **NIA** if an alternative A_k that is placed last in all voters' preference orderings — that provides utility of zero for all voters if utilities are standard vonNeumann-Morgenstern utilities scaled from zero to one — is a likely winner in equilibrium.

[4] They also examine a candidate-positioning game, which I do not discuss here.

For most of this section, the example I will use to investigate **NIA** is similar to the second of the Myerson-Weber examples, except that there is *no support whatsoever* for the third candidate. That is, there are three candidates, 1, 2, and 3 and two types of voters, A and B. Their preferences are defined as follows:

Voter Type	Utility Vector	Proportion of Electorate
A	$u^{A} = (10, t, 0)$	$f(u^{A}) = 0.50$
B	$u^{B} = (t, 10, 0)$	$f(u^{B}) = 0.50$

It is assumed that $0 < t < 10$. That is, (on average) 50% of the electorate prefers alternative 1 (utility of 10), but would rather get alternative 2 (utility of t) than 3 (utility of 0). Conversely, 50% of the electorate has a preference for alternative 2, but would still rather have alternative 1 than alternative 3. This is fairly typical of an election in a two-party system, where only two candidates / parties are taken seriously, but there may be other alternatives on the ballot.

The point of this exercise is not to model some real-world situation, however, but to identify any systems that are incapable of preventing alternative 3 from winning. That is, if 3 is as or more likely to win as alternatives 1 and 2 under voting system S, then system S violates **NIA**. I would argue further that if S violates **NIA**, S is useless. Systems that violate **NIA** allow alternatives to be elected no matter how universally disliked, merely by appearing on the ballot. It will turn out that a great number of the systems we have been discussing in this chapter, Borda among them, are useless in just this way. For those that like to hear the punch line first, the results of the analyses I discuss in the next few sections are shown in Table 4.2. Again, most voting systems violate **NIA**.

4.2.4 Scoring Systems

In this subsection, I focus on scoring systems. Excluding approval voting, this is a slightly broader set of voting systems than those discussed by Myerson and Weber, but a set that requires no modification of their equilibrium concept. We will here characterize a three-alternative scoring system as one in which the voter casts a ballot consisting of three numbers, 1, w, and 0, indicating the points to be given to the alternatives ranked first, second, and third, respectively. The value of w determines the specific voting system. If w=0, the system is SVP. If $w = 1$, the system is APV. If $w = 1/2$, the system is Borda. So, we have added APV and many intermediate systems to the Myerson-Weber list.

Rational behavior in a scoring system is straightforward. Expected utility is maximized by ranking alternatives according to what Myerson and Weber call the *prospective rating* of each alternative. The prospective rating of alternative j is simply for voter i:

$$r_j^i = \sum_{k \neq j} q_{jk} \left(u_j^i - u_k^i \right)$$

Each term in this sum is just the utility difference between alternative *j* and some other alternative, multiplied by the probability that these two are tied. For Borda, and other systems where 0<*w*<1, the alternatives are ranked by prospective rating. For SPV, the equivalent is simply to vote for the alternative with the highest prospective rating. For APV, the equivalent is simply to vote *against* the alternative with the highest prospective rating.

For the scoring systems, I will simplify matters and examine the specific case of t = 9. That is, here, all voters have only slight preferences for their first choice over their second, and much prefer their second to alternative 3. With this assumption, the prospective ratings for A-type voters in our example are as follows:

$$r_1^A = q_{12} + 10q_{13}$$
$$r_2^A = -q_{12} + 9q_{23}$$
$$r_3^A = -10q_{13} - 9q_{23}$$

The relationships among these can be displayed on a simplex as in Figure 4.10. The vertices indicate a tie probability of one for a specific pair of alternatives. There are two lines indicating ties between the prospective ratings of alternatives 1 and 2 and between those of 2 and 3. There is no distribution of tie probabilities that gives A voters the same prospective ratings for 1 and 3. These lines demarcate three regions of the space where A voters will rank the alternatives in their ballots 1>2>3, 2>1>3, or 1>3>2. Note that along these lines, A voters may mix (or split among themselves in arbitrary proportions) between the strategies allowed on either side of the line.

Proceeding with a similar (symmetric) exercise for type B voters, we are left with a situation as shown in Figure 4.11. There are now six regions of the space where we have specific pairs of pure strategies for the two types of voters. Mixtures of these strategies are allowed along the lines (and points) that separate the regions. Figure 4.11 also indicates the expected scores for the alternatives in each of these regions. What we now have are the strategies and outcomes for any set of beliefs. To find equilibria, we need now to determine which of these beliefs are consistent with the ordering condition.

Consider first region III in Figure 4.11. Here, everyone votes their sincere preference ordering and the expected scores are $[(1+w)/2, (1+w)/2, 0]$. Alternatives 1 and 2 are tied (in expectation) for first. The only belief consistent with the ordering condition is then q_{12}=1, but that vertex is not in region III. This yields the rather stark conclusion that *sincere voting is not in equilibrium under any scoring rule* (with plurality rule providing a partial exception discussed below).

Our equilibria must be elsewhere. It is easy enough, as well as instructive, to do this by brute force, so let's look at each of the other five regions individually. In region I, the alternatives will have expected scores of [1, *w*, 0], indicating that for *w* > 0 — any rule except plurality rule — the alternatives will be ranked 1>2>3. If that is true, then only a belief of q_{12}=1 is consistent. That is not true in region I, so there is no equilibrium in this region for any rule with *w*>0. (Beliefs would have to be in the nonadjacent region VI). If w =0 — SVP — then we can have any set of beliefs along the edge of region I where q_{23}=0. All of these are plurality rule equilibria in which 3 is expected to be more competitive with 1 than is 2, so all voters vote for 1. A symmetric set of equilibria exists under plurality rule only along the edge of region II, where all voters vote for alternative 2. So, for scoring rules other than plurality, there are no equilibria in which all voters coordinate and vote the same way (with alternative 3 last); for plurality, there are equilibria in which all voters vote for alternative 1 or for alternative 2.

In region IV, A-type voters elevate alternative three in their ranking to second. For rules other than plurality, this yields expected scores such that $s_1>s_2>s_3$, which again would require q_{12}=1. For plurality, it is not much different: **s** = [1/2, 1/2, 0], and again a requirement of q_{12}=1. This is not in this region so this is not in equilibrium. By symmetric reasoning, the same is true in region V. So, there are no equilibria, under any scoring rule, in which only one type of voter turkey-raises, while the other type votes sincerely.

In region VI, *both* types of voters "turkey-raise": elevating alternative three to second in their ranking. The resulting expected score vector is **s** = [1/2, 1/2, w]. For plurality and rules sufficiently close to it — specifically for rules with w<1/2 — alternative three finishes third and beliefs are q_{12}=1. This vertex *is* in the region, so there is an equilibrium for any rule with w<1/2, including plurality rule, with equilibrium beliefs of q_{12}=1 and in which alternative 1 and 2 are both expected winners. Note that this is not generally a sincere voting equilibrium, because everyone has lifted alternative 3 in their ranking from third to second. Under plurality rule, however, this simply means voting for the sincere first choice (1 for A types, 2 for B types), and this is the exception mentioned above.

Whether we call this "sincere" or not is irrelevant. The important finding is then that s*coring rules defined by* $w \in [0,1/2)$ *, including plurality rule, satisfy **NIA***.[5] That is, under plurality rule and scoring rules sufficiently close to plurality rule, an alternative cannot win simply by getting on the ballot. It should be noted, however, that under plurality rule there are multiple equilibria, so there are coordination problems to be solved to reach a particular one. It should also be noted that, although alternative three does not win, it still receives second place votes from all voters despite being universally viewed as worst. There are almost no examples of such voting systems in the real-world. One near-example is the voting system of Nauru, a scoring system with a point vector of [1, 1/2, 1/3, …, 1/K].[6] Translating to our scale for three alternatives, the Nauru point vector would be [1, 1/4, 0], so it falls in this "near-plurality" category. Indeed, what we observe is that every candidate in every district in every election receives a substantial number of points. It is possible that Nauru voters have never under any circumstances thought that anyone running for office was completely undeserving; it seems more likely that we are observing the predicted (if inconsequential for candidate selection) turkey-raising.

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[5] I have only shown this to be true in this example, of course, although I can assert with confidence that it is generally true.

[6] It is only a "near-example" because it is a multiple winner system, electing the two or four highest ranked alternatives. See the discussion in Chapter 7.

Now let's continue searching region VI. For w=1/2, the alternatives have the same expected values, requiring q_{12}=q_{13}=q_{23}, the center of the simplex (in region III). For rules with w>1/2, alternative 3 finishes first, requiring q_{13}=q_{23}=1/2, the midpoint of the opposite edge of the simplex, also in region III.

So at this point we have three sets of equilibria under plurality rule and one equilibrium for rules in which w<1/2. We have exhausted all of the regions of the space and found no equilibrium for Borda (w=1/2), APV (w=1), or any of the rules with $1/2 < w < 1$. These equilibria must, however, exist. We need to looks for them along the borders of the belief regions, where the types may be mixing their choice of voting strategies.

In regions I, II, III, IV, and V, the results lead to beliefs at the q_{12} vertex in region VI. In region VI, the results lead to beliefs in region III. The equilibrium is the point where these pulls on the beliefs just even out, the point E on Figure 4.11 where the beliefs are q_{12} = 28/30 and $q_{13} = q_{23}$ = 1/30. At E, voters can mix among the behaviors of the adjacent regions. Specifically, each voter votes sincerely for their first choice, but there only a proportion a of each type votes sincerely for their second choice over their third, with the remainder turkey-raising alternative 3 to second place. To have beliefs at this interior point the expected scores of the alternatives must be identical (otherwise one or more of the beliefs would have to be zero). The value of a is a function of the voting rule in place:

$$a = \frac{1}{3}\left(2 - \frac{1}{w}\right), \text{ for } \frac{1}{2} \le w \le 1$$

Under Borda rule, a = 1. Thus, A-types all vote 1>3>2 and B-types all vote 2>3>1. Under APV, $a = 1/3$. One-third of each type votes sincerely against alternative 3, two-thirds of A-types vote against alternative 2, and two-thirds of B-types vote against alternative 1.[7]

In both cases, and all rules in between ($1/2<w<1$), every alternative receives the same number of points and every alternative is a likely winner. That is, for these rules, there is just enough turkey-raising to make the expected point totals of all alternatives exactly equal. Moreover this is the *only* equilibrium under these rules. This leads to the conclusion that *all scoring systems defined by* $w \in [1/2, 1]$, *including Borda rule and anti-plurality voting, violate* ***NIA***. No matter how bad an alternative is universally agreed to be, if it is allowed on the ballot under Borda rule or APV, it is as likely to win as any other. Not only is Borda incapable of utility maximization, but it *guarantees* that the expected utility is just the average of all possible outcomes. It is no different than choosing randomly among the alternatives, regardless of how poor any of the alternatives might be.

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[7] Note that the similar example in Myerson and Weber (1993) yields multiple — albeit very similar — equilibria. In that case, the multiplicity is created by the arbitrary choice of strategy by the C-type voters (supporters of candidate 3) whose actions are then offset exactly by A&B-types. These equilibria are, in every substantively important way, identical. In any case, that is not an issue in this example.

Moreover, these properties are robust to changes in the parameters. We can remove the symmetry in preferences for types A and B and the beliefs move slightly, but the outcomes remain unchanged.[8] We can add further irrelevant or near irrelevant alternatives and the basic outcomes do not change.[9] We can add some small amount of support for alternative three and the basic outcomes do not change (as seen in the Myerson-Weber example).

4.2.5 Approval Voting

Approval voting is slightly different from the scoring systems. Here the voter is not choosing a rank ordering of the alternatives, but choosing which alternatives will receive a vote and which alternatives will not. In this context, utility maximization requires each voter to give votes to all alternatives with positive prospective ratings and to deny votes to those with negative prospective ratings. A quick glance at the definitions of the prospective ratings for our present situation makes it clear that each voter will always vote for their favorite (1 for A-types and 2 for B-types) and against their least favorite, alternative 3 for everyone in this case.[10] The key issue is then to identify the circumstances under which each type of voter also votes for their second choice.

We can draw a figure analogous to Figure 4.11, but that would be overkill here. Note that since (a) no one ever votes for alternative 3, and (b) alternative 1 and alternative 2 must always receive some votes regardless of the beliefs, the ordering condition requires that we cannot have any beliefs in which $q_{13}>0$ or $q_{23}>0$. The only allowable beliefs are then $q_{12}=1$. It then follows immediately that each voter has a positive prospective rating for only their top-ranked alternative, each voter single-votes, and the expected scores of alternative 1 and 2 are identical (and alternative 3 cannot win). *Approval voting satisfies **NIA**.*

This equilibrium looks superficially identical to the symmetric plurality rule equilibrium, but there are important differences between approval voting and plurality rule. First, the approval voting equilibrium is unique — no coordination problems need be solved to reach this equilibrium rather than another. Plurality rule commonly has such multiple equilibria; the inherent coordination problems (and related issues in similar voting systems) are the main issue explored by Cox (1997). These become more problematic as we increase support for the third alternative (contrast the equilibria here with those of

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[8] If we make type A larger than type B, we will see alternative 1 as the only likely winner in the $q_{12}=1$ equilibrium under plurality rule (rather than both 1 and 2), and vice versa.

[9] For a rule to be "sufficiently near" plurality rule to sustain the plurality rule equilibrium at $q_{12}=1$, the weights for intermediate rankings must average less than 1/2. An example of such a rule might be something like an American college basketball poll: rank the top 25 of these 200+ teams. Most of the "candidates" receive scores of 0, making such a rule "closer" to plurality than to Borda. (The polls are not used for selecting a single winner of any sort, so the strategic incentives discussed here to not necessarily apply.)

[10] These have long been known to be (weakly) dominant strategies under approval voting (Brams and Fishburn 1983).

Myerson and Weber's examples). Second, everyone single-votes for their favorite only because of the ability to double-vote for them (and the refusal by all voters to vote for their least favorite). In other situations (like those explored by Myerson and Weber), this yields preferable outcomes.

Note also that the nice properties of approval voting depend on there being only two levels of approval. These properties disappear when we allow partition into three groups or more. Consider Felsenthal's (1989) approval / disapproval voting, in which voters identify an "approved", a "disapproved", and, by default, an "other" group of alternatives. If q_{12}=1 as above, voters can do better than simply approving of their favorite. They can also disapprove of the main competitor. As discussed by Cox, the general logic here is simply that voters have an incentive to offer ballots that maximally separate the leading competitors. This requires turkey-raising under Borda and, with three partitions available rather than two, it requires it under approval / disapproval voting. *Approval / disapproval voting (and other g-partitioning rules with g>2) violate **NIA**.*

4.2.6 Paired Comparison Systems

The Myerson-Weber model works for systems that can be characterized as aggregating preference *vectors*: we consider the impact in some decision vector or moving alternatives up and down in a ballot vector. Most of the voting systems described in this chapter, however, are based on preference *matrices* — built on paired comparisons among the alternatives. The Myerson-Weber model does not extend directly to this set of systems and it is necessary to develop an analogous model; I refer herein to this analogous concpet as "paired comparison voting equilibria".

In this new model, the ballots — the strategy choices — available to each voter are general ordinal preference matrices. That is, if K alternatives are on the ballot, each voter expresses an ordinal preference for each of the $1/2K(K\text{-}1)$ pairs of alternatives. For the moment, let's just assume that we are looking at voting systems with no individual preference processing and simple averaging to a collective preference matrix, from which we can determine a winner without further reference to the individual ballots. This class of voting systems would include Borda, Simpson, Copeland, exponentiated versions of Borda or Copeland (including second-order Copeland and Kendall-Wei), minimum coercion systems (Kemeny, Slater, Dodgson, “closest majority order” / Black), and Nanson. We can also look at majority-weighting systems, like the voice of reason, with a minor additional assumption.

The essence of this model and its differences from Myerson-Weber is as follows.

1) In the paired comparison model, collective preferences are summarized by a collective preference matrix. In the Myerson-Weber model, they are summarized by a vector (of scores).
2) Under any given system, there are distinct "decisive outcomes", outcomes of the voting that imply a particular winner or winners for a distinct reason. Under Myerson-Weber, the decisive outcomes are just of the form "alternative 1 got the most points / alternative one wins". Under paired comparisons, they might be of the form "there is a majority cycle / all three altematives tie for the win" (under Copeland, for example).

3) Each system has a set of "pivot outcomes", outcomes that represent a "tie" between two decisive outcomes with different winners or sets of winners. Under Myerson-Weber, pivot outcomes are just ties in the scores of two alternatives. Under the new model, they might take the form of a "tie" between "alternative 1 is a Condorcet winner / alternative 1 wins" and "there is a majority cycle / all three alternatives tie for the win". An outcome on the border of two decisive outcomes does not constitute a pivot outcome unless the set of winners is different in the two decisive outcomes.

4) We view each pair of alternatives separately for the purpose of identifying voting strategies, conditional pivot probabilities, and prospective ratings. For example, a voter examines alternatives 1 and 2 to decide which to vote for on that particular pairing. She determines the pivot outcomes that can be affected by a vote on the (1, 2) pair and her beliefs (probabilities) of reaching each, given that some pivot is reached. These probabilities and the utility differentials on the decisive outcomes involved generate a prospective rating for each alternative, and the voter votes for the one with the higher rating. She does this for each of the $1/2K(K\text{-}1)$ pairs.

5) We again have an ordering condition in which the expected outcome from strategy choices must generate results consistent with the belief probabilities. A voting equilibrium is reached when strategies and beliefs are consistent with one another.[11]

It is most useful to explicate this model by looking at a few examples. Tables 4.3, 4.4, and 4.5 display (partial) information about the calculation of such equilibria for the Copeland, Borda, and Simpson models, respectively. Consider first Table (a) in each case, which display the "decisive outcomes". For Copeland, there are eight decisive outcomes: the distinct majority preference orders and the two cycles, clockwise (?) and counterclockwise (?). For Borda, there are only three: the score wins by each of the three alternatives. For Simpson, there are twelve: the six majority preference orders and then three outcomes for each majority cycle, one for each alternative's loss to be the smallest (the "weak link" that makes them the winner).

The "pivot outcomes" are listed in the (b) tables and consist of any outcome just on the edge of two of these, where the set of winners is not identical. So, for example a collective preference matrix of

$$C = \begin{bmatrix} 0 & 0.5 & 0.6 \\ 0.5 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

[11] The proof is just a fixed-point result, which I omit here.

is the pivot outcome 1? as listed in second row of Table 4.3b for the Copeland rule. There is a tie on the (1,2) pairing, while 1 has a majority over 3 and 3 has a majority over 2. So we are at a pivot between the decisive outcome 1? 3? 2 and the counterclockwise cycle, 1? 3? 2? 1 or "? ". The third column of table 4.3b indicates the pair where a change in votes changes the outcome, in this case from a win by alternative 1 to a tie among all three.

Note that even with *D* decisive outcomes, there are fewer than 1/2*D*(*D*-1) pivots. Some decisive outcomes are not adjacent. For example the outcome 1? 2? 3 is adjacent to 2? 1? 3, but not to 2? 3? 1. Moreover, some adjacent decisive outcomes produce the same winners (e.g., 1? 2? 3 and 1? 3? 2). Note also that the Simpson rule has 27 pivot outcomes in the example, nine for each pair of alternatives as pivots, but for space reasons I have only listed those determined by the (1,2) pair.

The last two columns of the (b) tables indicate the marginal differences in utility, for each type of voter in our example, of casting just this pivotal vote for the first member of the pivot pair. These in turn help define the prospective ratings for each pair for each voter, as listed in the (c) tables for Copeland and Borda (prospective ratings for Simpson not shown). These define voter strategies under the voting game.

For each pair, the voter assumes that one of the pivot outcomes determined by that pair has occurred. She has beliefs (the "q"s over these outcomes). If the expected utility differential (where the probabilities are these conditional belief probabilities) is greater than one, she votes for the first member of the pair; if it is negative, she votes for the second.

As with Myerson and Weber, what drives equilibrium is the ordering condition. Again, this is the requirement that in equilibrium, a belief which supports the equilibrium cannot be positive unless it is (among the) closest to the actual outcome. Let's consider our example and these three voting rules.

Consider first the Copeland rule. Each voter has three choices to make, but dominant strategies in some cases. Regardless of tie probabilities and the value of t, both types of voters vote for their first choice over their third in all circumstances (1 over 3 for A-types and 2 over 3 for B-types). If $t > 5$, then both types also vote for their second choice over their third, regardless of tie probabilities. The only issue then is whether they vote for their first choice over their second. If they do, then the expected collective preference is as shown in Figure 4.12a. This yields beliefs of q_{12}=1, which are consistent, so this is an equilibrium. If they both do not, this again yields beliefs of q_{12}=1, but all voters are acting opposite their preferences, so this is not in equilibrium. The other possibility is that they all coordinate — all voting, for example, for 1. This yields a unanimous ordering of 1? 2? 3, implying again q_{12}=1, and again B-type voters are voting against their preferences. So, for $t > 5$, sincere voting is the unique equilibrium, and the irrelevant alternative is not elected.

Consider, however, the possibility that $t < 5$. Now each voter prefers the creation of a cycle, and a three-way lottery among the alternatives, to outright selection of alternative 2. This means that voting for a second choice over the third is no longer a dominant strategy. Consider again the possibility that everyone is sincere (Figure 4.12a). Now the closest pivot outcome controlled by the pivot pair (2, 3) is that between a cycle (?) and an outright win for 2 (2? 1? 3). A-types would now prefer to take their chances with the cycle than to get alternative 2 for sure. So, sincerity is not in equilibrium here; they have an incentive to vote for 3 over 2. A similar circumstance applies to the B-types in their decision on the (3, 1) pair. Sincere voting is not an equilibrium for $t < 5$.

What if they all turkey-raise? Then the expected outcome is as shown in Figure 4.12b. Consider the possibility that everyone turkey-raises. This all-way tie of all alternatives makes our job very easy. All pivots are equidistant from the expected outcome and all beliefs are consistent with the ordering condition. In fact, for $t < 5$, turkey-raising (with a variety of beliefs) is the *only* equilibrium, and all three alternatives are likely winners. *The Copeland rule violates **NIA***.

As a check on the paired comparison model, let's see if it yields the same result as under Myerson-Weber for the one system that is both a paired comparison and scoring system: Borda. Table 4.4 shows the decisive outcomes, the pivot outcomes, and prospective ratings for Borda for our example. The only dominant strategy is in the pairing of first and last choices, where again no voter ever has the incentive to vote other than sincerely on that pair. There are beliefs, however, that allow switching the top two or bottom two preferences. As before, if people vote sincerely then the expected collective preference is consistent only with beliefs of q_{12}=1. These beliefs are consistent only with turkey-raising of third choice over second. Sincere voting is not in equilibrium. The unique equilibrium is, as before, the turkey-raising one. With all voters turkey-raising (Fig 4.12b), we have ties in expectation for all pairs of alternatives. This, in turn, is consistent with beliefs that allow for turkey-raising. We reach the identical conclusion using both models, which is exactly what we should expect.[12]

Table 4.5 shows the decisive outcomes and some of the pivot outcomes for the Simpson rule. There are 27 pivot outcomes in total, so the full listing of those and the subsequent prospective ratings is quite tedious and not particularly enlightening. We can get right to the point by examining the two cases of sincerity and turkey-raising that continue to arise. Consider again the expected collective preference under sincere voting. The closest pivot controlled by the pivot pair of (2,3) is that between ? 1 (a cycle with 1 having the smallest loss, to 2) and "21" (2? 1? 3).[13] An A-type voter, who prefers 1 to 2 should then vote for 3 over 2 on the (2,3) pairing. Similarly, B-types should vote for 3 over 1 on the (3, 1) pairing. Sincerity is not in equilibrium. As we have found with other systems, the unique equilibriumis the turkey-raising

[12] Note that other scoring systems, such as plurality, distort preferences over pairs in different ways depending on the rank of the alternatives involved. These systems cannot be analyzed using the paired comparison model. Borda is a common link between them, but neither model is in a technical sense a "generalization" of the other.

[13] This is equidistant to the border between "12" and "13", which is not a pivot outcome, because 1 wins in both cases.

one (again, beliefs as shown in Figure 4.12b are consistent with beliefs that support the behavior). Again, all three alternatives are likely winners and *the Simpson rule violates **NIA***.

We can repeat this exercise for all of the paired comparison systems, although some are even more complex. The Nanson rule is among the most complex. Votes on a particular pivot pair — let's use (1,2) as an example — can have two basic effects. First, the majority relation between 2 and 1 may change. This matters only if alternative three has (and continues to have) the lowest Borda count and is eliminated; otherwise the majority relation between 1 and 2 is irrelevant. Second, votes on (1,2) can change the Borda counts of both alternatives and, in turn, determine which of the three alternatives is eliminated. Even more so than Simpson, determining which pivot is closest can be complex. Finding equilibria is then considerably more difficult than under previously discussed systems, because the ordering conditions are so intricate and interdependent. That leaves opportunity for some interesting investigation of Nanson for some ambitious graduate student. For present purposes, however, we have all we need to be able to look at the irrelevant alternatives example.

If we have sincere voting (Fig. 4.12a), alternative 3 is eliminated and there is a tie in expectation for alternatives 1 and 2. The pivot condition on the (1,2) pairing is then obvious and each type should vote for their sincerely preferred alternative of the two. The closest pivot on the (3, 1) pair is a bit less obvious. If the value of c_{13} were to decrease from its expected value of 1, the outcome would not change until it reached a value of $c_{13} < 0.25$. At that point, alternative 1 has the lowest Borda count and is eliminated; now alternative 2 has a majority over alternative 3 and is declared the winner. So, the closest pivot on the (3, 1) pairing swings the winner from alternative 1 to alternative 2. A-type voters vote for 1 and B-types, preferring 2, vote for 3. By symmetry, the same occurs on the (2, 3) pairing and A-types vote for 3. Both are turkey-raising; is this in equilibrium? Absolutely. As with other systems, there are, in fact, infinitely many sets of beliefs that support this particular behavior and this particular expected outcome in which all three alternatives have an equal probability of victory. There are no equilibria in which alternative 3 is not a likely winner. Nanson violates **NIA**.

This exercise is easily repeated for the other paired comparison systems with identical results. Exponentiated versions of Copeland (second-order Copeland, eigenvector / Kendall-Wei) are identical to Copeland in the three-alternative case so that analysis applies in its entirety here: these systems violate **NIA** as well. The majority-based minimum-coercion systems are almost identical to Copeland. The unanimity-based minimum-coercion systems are almost identical to the Borda. Using converse or converse-consistent decision on the exponentiated systems or Simpson has no impact on the basic result.

This would get even more tedious if I continued in detail, so I will not. The conclusion is the same. All of these systems ask for and use complex preference information. All of them then give the incentive for voters to exaggerate their preference differences over the most competitive alternatives up to the point where it is unclear who the main competitors are. At this point all alternatives can win, including alternatives that have absolutely nothing to recommend them except their place on the ballot. All paired comparison systems violate **NIA**; all of them are useless.

It is reasonable to note that the likelihood of a turkey being among the winners varies across systems in other less symmetric examples. If there is a majority with the same preferences, for example, majoritarian systems (like Copeland) will always select the majority's first choice. So if we tweak the example just a bit, to give type-A 51% of the electorate for example, then systems like Copeland will select alternative 1 even in the face of strategic voters. Borda appears to be the least robust to such changes as turkeys win in a wide variety of preference configurations. There might then be some merit to mapping the relative probability of **NIA** violations in the paired comparison systems, but I do not pursue that here. I am sufficiently convinced that the presence of any such violations invalidates the use of the system, that I do not see much value in that.

4.2.7 Voice of Reason and Endogenous Voter-Weighting

Endogenous voter-weighting rules — I will look here only at majority weighting — can be analyzed by following logic very similar to that of Copeland, although they are not quite paired-comparison systems. We have the additional feature of needing the individual ballots themselves. One simple approach (which would not affect the paired comparison model results if added to the analyses of the previous sections) is to add an assumption that, with our infinite electorate, every possible ballot is offered by at least one voter. If a particular voting order is not in equilibrium, this one vote will have only infinitesimal weight. This help particularly with the analysis of the voice of reason: the highest agreement ordering will always be present. This way, we can assume that voters do not care at all about *being* the voice of reason, but only about affecting the majority preference matrix that determines the voice of reason ordering.

Now, let's examine the voice of reason by revisiting the Copeland discussion. If voters are sincere, then expected preferences are as in Figure 4.12a. The closest pivots on the (1,2) ordering are between 1? 2? 3, in which a voter with that ordering is voice of reason and alternative 1 is the winner, and 2? 1? 3, under which 2 is a winner. The closest pivot on (3, 1) is between 1? 2? 3 and ? ; the closest pivot on (2, 3) is between 2? 3? 1 and ? . Under Copeland, cyclic results produce a tie, with a random choice among the three alternatives and an expected value for each voter equal to the mean utility of the three alternatives. Under the voice of reason, a cycle produces a tie among all possible orderings, so each voter has an equal claim to being the voice of reason. This produces a lottery not among the alternatives, but among the voters, with the probability of any specific alternative being equal to the proportion of voters placing it first.

If everyone has voted sincerely, the lottery is then among just under 50% 1>2>3, just under 50% 2>1>3 and our assumed infinitesimal proportions of 1>3>2, 2>3>1, 3>1>2, and 3>2>1. This is then equivalent to a tie between alternatives 1 and 2, with expected value of the mean utility of those two. Type A voters will prefer alternative 1 to this tie, and this tie to alternative 2. So, Type A voters prefer to vote for 1 on the (1,3) pairing and 3 on the (2,3) pairing. The converse is true for Type B voters.

So again voters have an incentive to turkey-raise and we have contradicted the assumption of sincerity. As with all of the paired comparison voting systems, these behaviors are consistent with an expected outcome as shown in Figure 4.12b and supporting beliefs. Unlike the paired comparison systems, however, this does *not* produce alternative 3 as a likely winner. We still have only an infinitesimal portion of the electorate placing alternative 3 first, so there is zero probability of alternative 3 being a winner.

In fact, we can be more general than this. For an irrelevant alternative to be an expected winner under the voice of reason, it must be equilibrium behavior for at least some type of voter to place it *first* in a ballot ordering. While there are intermediate turkey-raising incentives, there is never any incentive to place such an alternative first, and we will always be able to eliminate irrelevant alternatives with the voice of reason. *The voice of reason satisfies **NIA**.*

The same is not true in general of the majority-weighting systems. Majority-agreement voting, for example, produces the same expected turkey-raising outcome. All voter types have equal majority weights and the winner is just calculated by a Borda count. Alternative 3 is as likely to win as the other three. *Majority-agreement voting violates **NIA**.*

4.2.8 Scoring-Elimination Systems

The other major remaining class of voting systems not yet discussed are the scoring-elimination systems, like alternative vote and Coombs. The elimination element prevents them from being analyzed according to the Myerson-Weber model. The elements of points distortion prevents them from being analyzed according to the paired comparison model. I will avoid a formal elimination model here, but we can use analogous logic to reach analogous results.

Let's think first about the alternative vote and whether sincere voting can be an equilibrium there. If voters are sincere, A-types voting 1>2>3 and B-types voting 2>1>3, then alternative 3 is eliminated and 1 and 2 are in a tie. We are at a pivot outcome and neither type has any incentive to switch their top two alternatives. Do they have any incentive to turkey-raise? If A-types turkey-raise and vote 1>3>2, then the outcome is identical: 3 is eliminated and a tie occurs between 1 and 2. No advantage there. The same occurs if both turkey-raise. Indeed, even if both turkey-raise, alternative 2 cannot win. *Alternative vote satisfies **NIA**.*

This looks very similar to plurality rule, so it is worth a brief sidebar to examine whether there are also equilibria in which both coordinate on one of the two main alternatives. For example, is it an equilibrium for all voters to vote 1>2>3. Here, we could eliminate either 2 or 3 in the first round and 1 wins in the second. B-types would prefer to have voted for 2 over 1 in case that was the last round matchup, so this is in fact not an equilibrium. Alternative vote does not have the multiple equilibria, and incumbent coordination problems, that plurality rule has.

The Coombs rule is the elimination version of ant-plurality voting and follows a similar logic. If voters are sincere, then 3 is eliminated and we have an expected tie between 1 and 2. A-type voters, however, can turkey-raise and create a tie for elimination between 2 and 3, and make the victory of alternative 1 more likely. Symmetrically, B-type voters can do the same. As with anti-plurality rule, the only equilibrium again is one in which all voters turkey-raise and all three alternatives have an equal likelihood of winning. *The Coombs rule violates **NIA**.*

We have already examined the Nanson rule as a paired comparison system and found it to violate NIA. The same is true if we look at it as an elimination system, as we should expect. The logic is parallel to that of the Coombs rule. If voters are sincere, then 3 is eliminated and 1 and 2 are in a tie for first. But each type can turkey-raise and create a competition for elimination and, unintentionally, a competition for winning. *Nanson violates **NIA**.*

The final system, runoff, is a bit different. It is the only system (I consider) that occurs in two rounds. Runoff being first-preference based, like alternative vote, it follows a similar logic, however. First, in our example there is no incentive to vote for alternative 3 instead of one's first choice in the first round. Doing so instead only raises the chances that one's favorite will not make the second round. Furthermore, if alternative 3 did make the second round, it would be unanimously defeated by any alternative it was paired with. In the second round, voters vote sincerely over the pairing that is left. this alone is sufficient to guarantee that *runoff satisfies **NIA***.

4.2.9 Preference Aggregation Redux

If we accept **NIA** as an axiom, we have only a handful of "acceptable" voting systems: plurality, near-plurality scoring systems, approval voting, alternative vote, and runoff, and the voice of reason. The common feature among these systems is the emphasis on first preferences. It is worthwhile, then, to examine whether there are substantial differences among them in their ability to select attractive alternatives that are not necessarily first preferences.

Consider the following example:

Voter Type	Utility Vector	Proportion of Electorate
A	$u^A = (10, 0, 9)$	$f(u^A) = 0.40$
B	$u^B = (0, 10, 9)$	$f(u^B) = 0.45$
C	$u^C = (3, 2, 10)$	$f(u^C) = 0.15$

There are three types. Type A ranks alternative 1 first, but is almost as happy with alternative 3. Type B — slightly larger than Type A — ranks alternative 2 first, but is almost as happy with alternative 3 as well. Type C ranks alternative 3 first, with a slight preference for 1 over 2. Note that alternative 3 is the clear social utility maximizer (as well as the Condorcet winner), but has only a small minority placing it first.

This is just the type of example that is often used in social choice discussions to indicate the inferiority of first-preference based systems. Alternative 3 should, by most social choice criteria, be the winner, yet first-preference based systems would appear to be poor at selecting it. Note also that while alternative 2 has the largest group preferring it most, it is the Condorcet loser: a majority prefers both 1 and 3 to it.

If voters are sincere, the systems vary in their ability to pick alternative 3. Under plurality rule, alternative 2 (the utility minimizer) is expected to tie. Under alternative vote or runoff, alternative 3 is eliminated and the votes of type C voters swing the result to alternative 1, a slight utility improvement. Under approval voting, *any* of the alternatives might win, depending on how many alternatives are approved by each group. It was this apparent indeterminacy that led to a debate — one which numbers among my personal candidates for most over-the-top academic debate ever — over whether approval voting constituted "an unmitigated evil" (Saari and Van Newenhizen 1988a, b; Brams et al. 1988a,b). Sincere voice of reason voters would produce a majority ordering of 3>1>2, exactly the ordering of type C voters, and alternative 3 would be the winner.

But, of course, there is no reason to presume sincerity. We should be concerned instead with how these systems perform in equilibrium. There are several obvious candidates for equilibrium behavior to be examined in each case: sincerity, coordination

Let's first examine plurality rule. There are Duvergerian equilibria in which 1 beats 2 (A- and C-types for 1, B-types for 2; $q_{12} = 1$), where 3 beats 1 (A-types for 1; B- and C-types for 3; $q_{13}=1$), and where 3 beats 2 (A- and C-types for 3, B-types for 2; $q_{23}=1$), the third candidate getting no votes in each case. There are no NonDuvergerian equilibria with all three receiving votes. We have multiple equilibria and in two of them, the utility-maximizing alternative 3 wins. The other equilibrium — electing second-best alternative 1 — seems by far the more likely, however. If beliefs develop from preelection polling and start at or near sincere beliefs, then 3 is unlikely to ever become a contender. Note, of course, that in any case we do not reach the *worst* outcome — alternative 2 — as we would have under sincere voting. Equilibria under near-plurality are identical, with ineffective turkey-raising by all voters, but identical outcomes.

Now consider approval voting. Recall that it is always (1) in equilibrium to approve of one's first choice, and (2) never in equilibrium to approve of one's last choice. The only question is who single-votes and who double-votes. It is not in equilibrium for all to single-vote. That would induce beliefs of $q_{12}=1$, in which case, C-types would have the incentive to double-vote, contra the assumption. There are, in fact only two equilibria. In the first, all C-types double-vote, $q_{12}=1$, and alternative 1 wins (with 2 in second). In the second equilibrium, all A-types double-vote, $q_{23}=1$, and alternative 3 wins (with 2 in second). So, approval voting also has multiple equilibria, but one less than does plurality rule. It is also true that the utility-minimizer (2) is never elected. The equilibrium in which 3 wins appears more plausible than those under plurality rule. If we imagine a process in which voters begin with sincere expressions of preference and then adjust, it plausible to imagine A- and B-type voters initially double-voting (since alternative 3 is almost as good as their first choice). B-types then adjust to single-voting and we reach our equilibrium.[14]

[14] It is worth noting in passing that the equilibrium behavior described by Brams and Fishburn (1983), in which voters approve of all alternatives with above average utility, is not in equilibrium in the Myerson-Weber framework. Here, that would require A- and B-types double-voting. Alternative 3 then wins (with 100% approval) and alternative 2 comes in second with 45%. But that implies $q_{23}=1$, in which case B-types would prefer to single-vote.

In summary, approval voting has multiple equilibria, one of which elects the second-best utility-maximizing choice. It is still plausible, however, that the second-best will be selected.

Both alternative vote and runoff follow logic similar to plurality rule, although here the equilibrium behavior can be characterized as "sincere". With sincere voting under alternative vote, alternative 3 is eliminated and alternative 1 wins in the final round. There is no incentive for A or B-types to vote for a different alternative first, as that only decreases the chance that their own favorite will make the final round or win if it does. So, the second-best choice will be chosen here (and never the worst). Runoff is identical to either alternative vote or plurality, depending on how we characterize preferences for going to a second round.

The voice of reason is intriguing. Under VOR, the unique equilibrium is the sincere voting one.[15] Moreover, this equilibrium selects a B-type voter as the voice of reason and alternative 3 is elected. This, interestingly, makes the voice of reason the only system guaranteed to select the utility maximizer (in this case). We can design examples where none of these picks the utility maximizer, so this should not be overdrawn.

It is also perhaps worth a brief sidebar to note how the Borda rule performs in this example. As might be expected, the only equilibria involve beliefs at an interior point of the simplex, where all three alternatives are expected winners. There are multiple equilibria involving different mixes of rankings by the different types, but the end result is as it (almost) always is under Borda. Every alternative has the same chance of winning and preference aggregation is no better than a random choice from among the alternatives.

This is, of course, nothing resembling a full analysis of the preference aggregation properties of all voting systems. It is informative, however. All of the **NIA**-compliant systems perform reasonably, avoiding election of the worst alternative in all cases. Only plurality, approval voting, and the voice of reason can elect the utility maximizer. In this example, three relative likelihood of doing so seems to be

[15] Again assume all preference orders and use the Copeland logic. Under sincere voting, $E(C_{\text{sincere}}) = \begin{bmatrix} 0 & 0.55 & 0.40 \\ 0.45 & 0 & 0.45 \\ 0.60 & 055 & 0 \end{bmatrix} \Rightarrow E(M_{\text{sincere}}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, B-type voters agree entirely with every majority and their first preference, alternative 3, is elected. The closest pivot outcome controlled by (2,3) is that between ? and 3? 1? 2, and all voters prefer their sincere choice. The closest pivot outcome controlled by (3,1) is between 3? 1? 2, and 1? 3? 2, and all voters prefer their sincere choice. The closest pivot outcome controlled by (1,2) is a little trickier. (1,2) alone controls only 3? 1? 2, and 3? 2? 1, which is not a pivot outcome since 3 wins in both cases. We must assume one of the other majorities also switches. The 3? 2 majority is smaller than the 3? 1 majority, so the closest pivot controlled by (1, 2) is that between ? and 2? 3? 1, and again everyone prefers their sincere choice.

highest in the voice of reason, second in approval voting, and third in plurality. In particular, the utility-maximizing outcomes under plurality involve what appear to be unlikely solutions to coordination problems. I return in the next chapter to another method of examining preference aggregation under these systems, which reaches similar conclusions.

Several points from the preference aggregation sections should be summarized here. First, most voting systems that have been considered in the social choice literature are essentially useless, if we demand of them some discrimination between social utility maximizers and minimizers in the face of strategic voting. The absolute worst voting system by this measure appears to be the one that is ideal in the face of sincere voters: the Borda rule. Indeed, it is the symmetry and responsiveness of the Borda rule that make it so susceptible to manipulation (to everyone's disbenefit). Since most voting systems have inherent in them only minimal distortions from this basic rule, most are only slightly less flawed. It appears that we must, in fact, accept some asymmetry — in particular an emphasis in some way on first preferences — and some nonpositive or even negative responsiveness, to have a system that produces even approximately desirable results. It turns out that only a handful of systems accomplish this in this setting. These are the three systems that are actually used in substantial real-world elections (plurality, alternative vote, and runoff) and three theoretical alternatives (near-plurality rule, approval voting, voice of reason). As a precursor to the discussions of Chapter 6 and beyond, it is also worth noting that the multiple-winner systems in use in the real-world are extensions of these same single-winner systems (most degenerate to plurality rule in the single-winner special case; single transferable vote degenerates to alternative vote; Kiribati's electoral system degenerates to runoff).

[Intervening Material Deleted]

4.4 Conclusions

This has been a complex chapter, so I will use this section to recap the highlights.

First, if we accept ordinal preferences as the inputs to a choice aggregation mechanism, the fundamental system — the one which involves every possible form of symmetry and maximum responsiveness — is the Borda rule. It has not been much mentioned here, but if we have other kinds of preference inputs, such as cardinal preferences, there are analogous fundamental systems, such as utilitarianism. If we have accurate preference information from our voters, Borda is by many measures an ideal voting system. This is closely related to the axiomatic characterization of Borda in Chapter 2.

Second, all other voting systems can be characterized by the specific ways in which they deviate from the perfect symmetry and responsiveness of Borda. Some — like approval voting — offer ballots / strategies that do not look exactly like ordinal preferences. Some — like plurality rules and alternative vote — distort the preference information over pairs of alternatives, giving greater weight to some pairs than others. Some — like Copeland — filter collective preferences to requantify paired comparisons. Some —

like the voice of reason — give different weight to voters depending on how much like other voters they are. Some — like Simpson and Nanson — use different decision rules. Some — like Kemeny and Slater — try to find unambiguous collective preference structures that are closest to the actual collective preference. The specific institutional variables that categorize an institution's deviations from the basic Borda structure are the source of particular properties of that institution.

Third, a system that is perfectly symmetric and responsive — like Borda — is highly manipulable and, more important, manipulable in very undesirable ways. Specifically, in most situations, strategic Borda voters are in equilibrium only at a point where all alternatives — no matter how good or bad — are equally likely to win. This was true in the Myerson and Weber (1993) examples and it is true here in the running **NIA** example. If voters are strategic, Borda is no more than a lottery among all the alternatives on the ballot, regardless of voter preferences for them. As an actual voting system then, Borda is not just less than ideal; it is useless.

Fourth, most voting systems are little better than Borda on this score. Most of the voting systems that have been discussed in the social choice literature can be characterized as "paired comparison" systems. In such systems, once we have complete information about the voters' preferences over every pair of alternatives — the collective preference matrix — we can determine the winner. These systems are, like Borda, completely symmetric. Only Borda is completely (positively) responsive; all of the other paired comparison systems are partially (nonnegatively) responsive. All of these violate **NIA** — can elect a completely unwanted alternative in equilibrium — with less responsive systems less susceptible to the problem. Copeland, for example, can eliminate "very irrelevant" alternatives.

Fifth, for a voting system to be useful in the face of strategic voters, it must differ substantially from the Borda "ideal" in specific ways. One important element is asymmetry in favor of first-preferences, as is the case with plurality rule, alternative vote, runoff, as well as (arguably) approval voting and the voice of reason. If a system does not have such an asymmetry, then alternatives that do not have much support can be used as strategic levers to undermine those that do. A second element is some lack of responsiveness. The most dramatic example is the voice of reason, which is negatively responsive. Under Borda, as voters become less like other voters, they become more influential; under the voice of reason, they become less influential. I illustrate this point more clearly in the next chapter. Alternative vote and runoff can also be negatively responsive (amonotonic) in certain circumstances. Plurality rule and approval voting are not negatively responsive, but are certainly blunt and unresponsive in important ways. In general, a voting system must only ask for limited preference information or it must ignore, or appear to act in opposition to, some of the information it is given. Otherwise its manipulability renders it useless.

Sixth, this inherent "uselessness" may be one real reason that the wide variety of theoretical voting systems remains, in Riker's words, "interesting but unused". While many have not been used at all, those that have, but which violate **NIA**, have been quickly abandoned. Consider, for example, the abortive attempts to use the Nanson rule, as catalogued by McLean (1996), or the disastrous consequences of adopting the Borda rule for voting on the highly politicized Booker prize, as reported by Treglown (1991).

Moreover, the real-world systems in the most important democratic setting — electing multiple winners — are also all based on first-preference-biased systems. I pursue this in much greater detail in Chapter 9, where I discuss the inherent boundaries to our ability to engineer democratic institutions to our whims.

Seventh, the ability of voting systems to aggregate information remains a largely open question. I can assert with confidence that the conventional wisdom on optimal systems for information aggregation is wrong, at least if we accept the possibility of strategic action. The Borda rule and other statistically-appropriate systems (Kemeny, Young, etc.) are no better than simple plurality rule when doing nothing but aggregating information. We know these rules are significantly inferior when voters have different preferences, so there is little to recommend them. Moreover, it is no defense that individuals will be too community-oriented when they have common values to vote strategically. As we saw in Chapter 3, it is the strategic voters who do *better* collectively (under an appropriate voting rule) than do sincere voters, because they can collectively get the statistics right. There is every reason to believe that continues to be true in a setting of choosing from among three or more alternatives.

In the next chapter, I use an arguably more complex democratic setting — choice from an infinite set of alternatives — to illustrate some of the key points that emerge in this chapter. In particular, it is important to understand the interplay of strategic manipulation and responsiveness in voting systems. if voting systems are to aggregate preferences in a way that is even remotely acceptable, they must ask for or respond to only certain types of information. In subsequent chapters, I turn to the important democratic setting of representation systems: choosing multiple winners.

Table 4.2 — Nonelection of Irrelevant Alternatives, Summary

System	NIA	Equilibrium Concept
Plurality (Scoring, $w = 0$)	✓	Myerson-Weber
Borda (Scoring, $w = 0.5$)	×	Myerson-Weber, Paired Comparison
Anti-Plurality (Scoring, $w = 1$)	×	Myerson-Weber
Scoring, $w < 0.5$	✓	Myerson-Weber
Scoring, $w \geq 0.5$	×	Myerson-Weber
Approval Voting	✓	Myerson-Weber
Approval-Disapproval Voting	×	Myerson-Weber
Simpson	×	Paired Comparison
Copeland	×	Paired Comparison
Second-order Copeland	×	Paired Comparison
Eigenvector / Kendall-Wei	×	Paired Comparison
r^{th}-order Copeland	×	Paired Comparison
r^{th}-order Borda	×	Paired Comparison
Converse Simpson	×	Paired Comparison
Converse r^{th}-order Copeland	×	Paired Comparison
Converse r^{th}-order Borda	×	Paired Comparison
Converse-consistent Simpson	×	Paired Comparison
Conv.-cons. r^{th}-order Copeland	×	Paired Comparison
Conv.-cons. r^{th}-order Borda	×	Paired Comparison
Slater	×	Paired Comparison
Kemeny	×	Paired Comparison
Dodgson	×	Paired Comparison
Closest Majority Order	×	Paired Comparison
Black	×	Paired Comparison
Elimination Copeland	×	Paired Comparison
Nanson	×	Paired Comparison, Elimination
Voice of Reason	✓[a]	Paired Comparison[b]
Majority-agreement Voting	×	Paired Comparison[b]
Alternative Vote	✓	Elimination
Coombs	×	Elimination
Runoff	✓	Elimination[c]

[a] With probability approaching one. See text.
[b] With additional assumption that all orders receive at least one vote. See text.
[c] As modified to allow for two rounds of voting. See text.

Table 4.3 — Determining Voting Equilibria Under the Copeland Rule

(a) Decisive Outcomes

Symbol	Outcomes	Winner(s)	A payoff	B payoff
12	1? 2? 3	1	10	τ
13	1? 3? 2	1	10	τ
21	2? 1? 3	2	τ	10
23	2? 3? 1	2	τ	10
31	3? 1? 2	3	0	0
32	3? 2? 1	3	0	0
?	1? 2? 3? 1	{1,2,3}	$(10+t)/3$	$(10+t)/3$
?	1? 3? 2? 1	{1,2,3}	$(10+t)/3$	$(10+t)/3$

"? " indicates majority preference.

(b) Pivot Outcomes

Symbol	Between	Pivot Pair	A Payoff Differential	B Payoff Differential
12	(12, 21)	(1, 2)	$10-t$	$t-10$
1?	(13, ?)	(1, 2)	$(20-t)/3$	$(2t-10)/3$
? 2	(? , 2)	(1, 2)	$(10-2t)/3$	$(t-20)/3$
23	(23, 32)	(2, 3)	t	10
2?	(21, ?)	(2, 3)	$(2t-10)/3$	$(20-t)/3$
? 3	(? , 31)	(2, 3)	$(10+t)/3$	$(10+t)/3$
31	(31 ,13)	(3, 1)	-10	$-t$
3?	(32, ?)	(3, 1)	$-(10+t)/3$	$-(10+t)/3$
? 1	(? ,12)	(3, 1)	$(t-20)/3$	$(10-2t)/3$

(c) Prospective Ratings

A votes for 1 over 2 if $q_{12}(30 - 3t) + q_{1?}(20 - t) + q_{?2}(10 - 2t) > 0$ $(q_{12} + q_{1?} + q_{?2} = 1)$
A votes for 2 over 3 if $q_{23}(3t) + q_{2?}(2t - 10) + q_{?3}(10 + t) > 0$ $(q_{23} + q_{2?} + q_{?3} = 1)$
A votes for 3 over 1 if $q_{31}(-10) + q_{3?}(-10 - t) + q_{?1}(t - 20) > 0$ $(q_{31} + q_{3?} + q_{?1} = 1)$

B votes for 1 over 2 if $q_{12}(3t - 30) + q_{1?}(2t - 10) + q_{?2}(t - 20) > 0$ $(q_{12} + q_{1?} + q_{?2} = 1)$
B votes for 2 over 3 if $q_{23}(30) + q_{2?}(20 - t) + q_{?3}(10 + t) > 0$ $(q_{23} + q_{2?} + q_{?3} = 1)$
B votes for 3 over 1 if $q_{31}(-3t) + q_{3?}(-10 - t) + q_{?1}(10 - 2t) > 0$ $(q_{31} + q_{3?} + q_{?1} = 1)$

Table 4.4 — Determining Voting Equilibria Under the Borda Rule

(a) Decisive Outcomes

Symbol	Outcomes	Winner(s)	A payoff	B payoff
1	1	1	10	τ
2	2	2	τ	10
3	3	3	0	0

(b) Pivot Outcomes

Symbol	Between	Pivot Pair	Payoff Differential	B Payoff Differential
12	(1, 2)	(1, 2)	$10 - t$	$t - 10$
23	(2, 3)	(1, 2)	$-t$	-10
31	(3, 1)	(1, 2)	10	t
12	(1, 2)	(2, 3)	$t - 10$	$10 - t$
23	(2, 3)	(2, 3)	t	10
31	(3, 1)	(2, 3)	10	t
12	(1, 2)	(3, 1)	$t - 10$	$10 - t$
23	(2, 3)	(3, 1)	$-t$	-10
31	(3, 1)	(3, 1)	-10	$-t$

(c) Prospective Ratings

A votes for 1 over 2 if $q_{12}(10 - t) + q_{23}(-t) + q_{31}(10) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$
A votes for 2 over 3 if $q_{12}(t - 10) + q_{23}(t) + q_{31}(10) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$
A votes for 3 over 1 if $q_{12}(t - 10) + q_{23}(-t) + q_{31}(-10) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$

B votes for 1 over 2 if $q_{12}(t - 10) + q_{23}(-10) + q_{31}(t) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$
B votes for 2 over 3 if $q_{12}(10 - t) + q_{23}(10) + q_{31}(t) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$
B votes for 3 over 1 if $q_{12}(10 - t) + q_{23}(-10) + q_{31}(-t) > 0$ $(q_{12} + q_{23} + q_{31} = 1)$

Table 4.5 — Determining Voting Equilibria Under the Simpson Rule

(a) Decisive Outcomes

Symbol	Outcomes	Winner(s)	A payoff	B payoff
12	1? 2? 3	1	10	τ
13	1? 3? 2	1	10	τ
21	2? 1? 3	2	τ	10
23	2? 3? 1	2	τ	10
31	3? 1? 2	3	0	0
32	3? 2? 1	3	0	0
? 1	1? ? 2? ? 3? 1	1	10	τ
? 2	1? 2? ? 3? ? 1	2	τ	10
? 3	1? ? 2? 3? ? 1	3	0	0
? 1	1? ? 3? ? 2? 1	1	10	τ
? 2	1? ? 3? 2? ? 1	2	τ	10
? 3	1? 3? ? 2? ? 1	3	0	0

"? " and "? ? " indicate majority preference. If both shown, "? " indicates the *smallest* majority.

(b) Pivot Outcomes [on pivot pair (1,2) only]

Symbol	Between	Pivot Pair	A Payoff Differential	B Payoff Differential
12	(12, 21)	(1, 2)	$10 - t$	$t - 10$
2? 1	(? 1, 23)	(1, 2)	$10 - t$	$t - 10$
2? 3	(? 3, 23)	(1, 2)	$-t$	-10
? 12	(? 1, ? 2)	(1, 2)	$10 - t$	$t - 10$
? 23	(? 3, ? 2)	(1, 2)	$-t$	-10
1? 2	(13, ? 2)	(1, 2)	$10 - t$	$t - 10$
1? 3	(13, ? 3)	(1, 2)	10	t
? 12	(? 1, ? 2)	(1, 2)	$10 - t$	$t - 10$
? 13	(? 1, ? 3)	(1, 2)	10	t